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Tsunami generation

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The problems of tsunami generation are treated by standard integral-transform and modified stationary-phase methods to yield asymptotic approximations to the surface disturbances. The effects of asymmetry are considered in a one-dimensional ocean. Series representations are used to produce sets of normal-mode oscillations in a two-dimensional ocean, and the magnitudes of the wave front and wave train are discussed in relation to the asymmetry of the generating region.

1. Introduction

A tsunami (sometimes called a seismic sea wave) in a seismically generated sea wave which often has catastrophic effects on near and distance coastal regions. It is largely a Pacific Ocean phenomenon, and for centuries its ravages have plagued Pacific civilizations; the first documented tsunami occurred in 173 A.D. (Iida, Cox & Pararas-Carayannis 1967). Japan, in particular, has suffered from these destructive waves but other Pacific regions have been attacked, while a large tsunami is usually observed throughout the whole Pacific Ocean.

To minimize the loss of life and property, the Tsunami Warning Service was established by the United States Coast and Geodetic Survey and this system has been successful in issuing advanced warnings of the approach of a destructive tsunami. However, the Warning System cannot provide a reasonable prediction of tsunami run-up heights, this information being essential for the determination of zones of evacuation (Hwang & Divoky 1970). Two factors are necessary for accurate prediction of tsunami heights: information on the magnitude and distribution of the ground displacements caused by an earthquake, and a method of determining the surface waves resulting from this motion of the sea floor.

This contribution addresses the second problem. However, it is recognized that determination of the actual ground motion is a very difficult problem requiring significant technological advances before accurate estimates can be obtained. Theoretical aspects of the wave generation problem reduce to the need to solve a Cauchy–Poisson problem, which is usually handled in a region of constant depth. Webb (1962), Van Dorn (1965) and Carrier (1971) have discussed the literature and compared results obtained for various simplified models of the sea-floor disturbance. Van Dorn (1965) has presented an analytic formulation based on the work by Kajiura (1963) for the distant surface amplitude due to an arbitrary sea-floor disturbance in water of constant depth. Hwang & Divoky (1970) have used numerical methods on the basic hydrodynamic equations in a study of the Alaskan tsunami of 1964. They also dealt with an elliptic generating region on the sea floor, a feature of earthquakes which has been discussed by Hatori (1970). An analytic study of the sea waves produced by a relatively simple elliptic generating region has been carried out by van den Driessche & Braddock (1972*a*).

All of the models discussed above lack flexibility in that they approximate only a few particular earthquakes and cannot be applied to more general bottom disturbances. In this paper, the bottom motion is described by series of orthogonal functions and the resulting surface disturbance is expressed as a set of normal modes, which are discussed in detail. The nature of the orthogonal functions which are used ensures that quite complicated bottom disturbances can be accurately represented by only a few terms in the series. The important effects of the asymmetry of the generating region are also discussed in some detail.

2. Formal solution of general problem

Consider an incompressible inviscid fluid of infinite horizontal extent and constant depth $h > 0$ which is bounded above by a free surface. The Cartesian co-ordinate system Ox, y, z is located on the bottom with Oz vertically up, and the free surface is at

$$z = \zeta(x,y,t) + h \tag{1}$$

for all t, where $\zeta(x,y,t)$ is the displacement of the fluid surface from its mean position. The fluid is assumed to be at rest for $t \leqslant 0$, implying that $\zeta = 0$ for $t \leqslant 0$. If the flow is assumed irrotational, the velocity potential $\Phi(x,y,z,t)$ satisfies Laplace's equation

$$\nabla^2\Phi(x,y,z,t) = 0 \tag{2}$$

in the region $x, y \in (-\infty,\infty)$, $z \in (0, \zeta+h)$, $t \in (0,\infty)$. The linearized boundary conditions are

$$\left.\begin{aligned} \Phi_{tt} + g\Phi_z = 0, \qquad \Phi_t = -g\zeta \quad \text{at} \quad z = h, \\ \Phi_z = F(x,y,t) \quad \text{at} \quad z = 0, \end{aligned}\right\} \tag{3}$$

where the subscripts represent partial derivatives. Here it is assumed that the sea-floor disturbance can be modelled as a velocity distribution $F(x,y,t)$, $t \in (0,\infty)$, applied to the water at $z = 0$.

The solution to the above Cauchy–Poisson problem can be obtained using standard transform techniques (van den Driessche & Braddock 1972*a*). Using a Laplace transform on t, giving the transform variable s, and Fourier transforms on x and y, giving transform variables k and l, the resulting surface displacements are given by

$$\zeta(x,y,t) = \frac{-i}{8\pi^3}\int_{-\infty}^{\infty}\int_{\xi-i\infty}^{\xi+i\infty} \frac{s\exp(i\mathbf{k}.\mathbf{r}+st)\hat{F}(\mathbf{k},s)}{\cosh(|\mathbf{k}|h)[s^2+g|\mathbf{k}|\tanh(|\mathbf{k}|h)]}\,ds\,d\mathbf{k}, \tag{4}$$

where $\mathbf{r} = (x,y)$, $\mathbf{k} = (k,l)$ is the two-dimensional wavenumber, $d\mathbf{k} = (dk, dl)$, the integral over s is an inverse Laplace transform and

$$\hat{F}(\mathbf{k},s) = \int_{-\infty}^{\infty}\int_{0}^{\infty} F(\mathbf{r},t)\exp(-i\mathbf{k}.\mathbf{r}-st)\,dt\,d\mathbf{r}. \tag{5}$$

The inverse Laplace transform may be evaluated by means of a suitable contour integral. Simple poles are located at $s = \pm i\sigma$, where

$$\sigma = [g|\mathbf{k}|\tanh(|\mathbf{k}|h)]^{\frac{1}{2}};$$

these poles on the imaginary axis in the s plane lead to surface wave motions which are purely oscillatory, and which represent propagating wave trains. For those $F(\mathbf{r},t)$ which represent realistic disturbances of the sea floor the transforms $\hat{F}(\mathbf{k},s)$ are such that all poles are to the left of the imaginary axis in the s plane. These yield solutions which decay exponentially in time and which are generally of little importance.

If only the residues from the poles at $s = \pm i\sigma$ are considered, the oscillatory surface disturbances are given by

$$\zeta(x,y,t) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{\exp(i\mathbf{k}.\mathbf{r})}{\cosh(|\mathbf{k}|h)}[\hat{F}(\mathbf{k},i\sigma)\exp(i\sigma t)+\hat{F}(\mathbf{k},-i\sigma)\exp(-i\sigma t)]\,d\mathbf{k}. \tag{6}$$

The terms $\exp[i(\mathbf{k}.\mathbf{r}\pm\sigma t)]$ represent propagating wave motions on the surface of the water, and the functions

$$\psi_{\pm} = \hat{F}(\mathbf{k}, \pm i\sigma)/8\pi^2\cosh(|\mathbf{k}|h) \tag{7}$$

are called the complex wave amplitudes. Explicit evaluation of the integrals occurring in (6) is generally very difficult and various approximate methods are used to obtain estimates of $\zeta(x,y,t)$. These methods and results will be considered in detail in later sections of the paper.

3. One-dimensional case

Now consider the corresponding problem with only one horizontal dimension in which there is no y dependence in the system (2) and (3). The condition on the applied velocity at the bottom is expressed in the form

$$\Phi_z(x,z,t) = F(x,t) \quad \text{at} \quad z = 0.$$

Then only one Fourier transform is required and the propagating surface disturbances corresponding to (6) are given by

$$\zeta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\exp(ikx)}{\cosh(kh)}[\hat{F}(k,i\sigma)\exp(i\sigma t)+\hat{F}(k,-i\sigma)\exp(-i\sigma t)]\,dk, \tag{8}$$

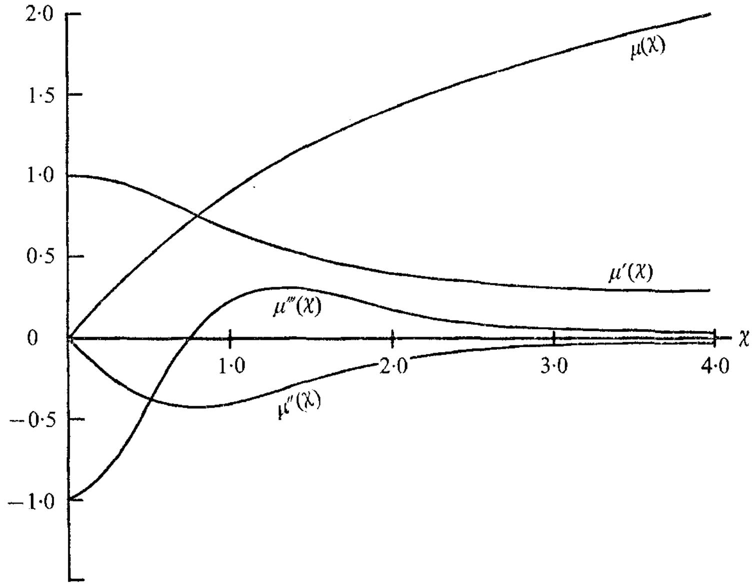

FIGURE 1. Graph of $\mu(\chi) = (\chi \tanh \chi)^{\frac{1}{2}}$ and its derivatives.

where k is now a one-dimensional wavenumber and $\sigma = [gk \tanh(kh)]^{\frac{1}{2}}$. The integrals occurring in (8) can be conveniently written in the form

$$I_n = \int_{-\infty}^{\infty} \frac{\hat{F}(k, (-1)^{n+1} i\sigma)}{\cosh(kh)} \exp[i(kx + (-1)^{n+1}\sigma t)]\, dk \qquad (n = 1, 2). \tag{9}$$

The application of the method of stationary phase (Copson 1965, p. 33) to the I_n is facilitated by setting

$$\xi = x/t, \quad \phi_n(k, \xi) = k\xi + (-1)^{n+1}\sigma(k) \quad (n = 1, 2), \tag{10}$$

where the $\phi_n(k, \xi)$ are the phase functions of the integrals. The points $k = k^*$ at which the phase functions are stationary are determined by the conditions

$$\partial\phi_n/\partial k = 0$$
$$= \xi + (-1)^{n+1} d\sigma/dk,$$

thus

$$x/t = (-1)^n C_g(k), \tag{11}$$

where $C_g(k)$ is the group velocity of the wave motion. The term $(-1)^n$ indicates the direction of propagation, to $x = \pm\infty$ for $n = 1$ and 2, respectively. Let $\chi = kh$ and

$$\mu(\chi) = (\chi \tanh \chi)^{\frac{1}{2}}, \tag{12}$$

then

$$C_g(\chi) = (gh)^{\frac{1}{2}} \mu'(\chi). \tag{13}$$

$\mu(\chi)$ and its first three derivatives are given in figure 1. The function $\mu'(\chi)$ is a monotonic decreasing function which has a maximum value of 1 at $\chi = 0$ and tends to zero as $\chi \to \infty$.

For $|x| > t\,(gh)^{\frac{1}{2}}$, there is no value of k^* satisfying (11), and hence no stationary-phase point. Physically, the surface disturbance has yet to penetrate this region since

$$I_1 = I_2 \approx 0, \tag{14}$$

to order t^{-2}. As $\chi \to \infty$, ξ, $\mu'(\chi)$ and the higher derivatives tend to zero. For large t, and $x \approx 0$, $I_n \to 0$ and the surface returns to its undisturbed state. For intermediate values, there is one stationary-phase point k^*, and the approximate values of the integrals are, where $\sigma^* = \sigma(k^*)$ and $\chi^* = k^*h$,

$$I_n \approx \frac{(2\pi)^{\frac{1}{2}}\,\hat{F}(k^*, (-1)^{n+1}i\sigma^*)}{\cosh(k^*h)} h^{-\frac{3}{4}} g^{-\frac{1}{4}} |\mu''(\chi^*)|^{-\frac{1}{2}} t^{-\frac{1}{2}} \exp\{i[k^*x + (-1)^n(-\sigma^*t + \tfrac{1}{4}\pi)]\}. \tag{15}$$

The I_n thus represent propagating sinusoidal surface waves which decay as $t^{-\frac{1}{2}}$.

The approximations (15) are not valid near $k^* = 0$, since near this point $\partial^2\phi_n(k,\xi)/\partial k^2 \approx 0$ (Copson 1965). Near the wave front at $k^* = 0$, the asymptotic values of (9) are given by

$$I_n \approx t^{-\frac{1}{3}}\Gamma(\tfrac{1}{3})\,h^{-\frac{5}{6}} g^{-\frac{1}{6}} 2^{\frac{1}{3}} 3^{-\frac{1}{6}} \hat{F}(0,0), \tag{16}$$

where $\Gamma(\frac{1}{3})$ is the gamma function with argument $\frac{1}{3}$. Thus the wave front decays more slowly than the main wave system and is more important than the following waves.

The approximations (15) and (16) are valid provided that the function $\hat{F}(k^*, (-1)^{n+1}i\sigma^*)$ is not zero. If this function is zero, other approximations can be obtained (van den Driessche & Braddock 1972*b*), and they are generally of lower order in t. Realistic forms of $F(x,t)$ are such that (15) remains a valid approximation to (9) for $k^* \neq 0$, but it is easily shown that $\hat{F}(0,s)$ is zero for asymmetric disturbances, i.e. if $F(x,t) = -F(-x,t)$, then $\hat{F}(0,s) = 0$. In such cases,

$$I_n \approx t^{-\frac{2}{3}}(-1)^n i\Gamma(\tfrac{2}{3})\,3^{\frac{1}{6}} 2^{\frac{2}{3}} g^{-\frac{1}{3}} h^{-\frac{2}{3}}\,(\partial\hat{F}/\partial k)_{k^*=0},$$

and this is of lower order in t than the approximation above. Thus for an asymmetric generating region, the wave front decays more rapidly than the body of the wave given by (15). This represents a change in the relative importance of these two sections of the tsunami since, for other generating regions, the front given by (16) decays more slowly than the main wave system given by (15).

A physical explanation is easily given in terms of monopole and dipole sources of water waves. The general case of a non-asymmetric generating region can be thought of as a monopole source of water waves where the wave front is dominant. The asymmetric case is represented by a dipole source comprising positive and negative monopole sources. The dominant sections of the wave fields from the component monopole sources, that is the wave fronts, effectively cancel. The body of the wave system is then dominant.

4. Two-dimensional case

The two-dimensional case has been handled by several authors; references to the literature are given in the introduction. Some of these attacks use polar space co-ordinates coupled with an assumption of no angular dependence and certain features of the solution are lost by this assumption. Here the general Cartesian form (6) will be considered without this simplifying assumption.

Equation (6) can be expressed in the form

$$\zeta(x,y,t) = +(8\pi^2)^{-1}(P_1+P_2), \tag{17}$$

where
$$P_{1,2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{F}(k,l,\pm i\sigma)}{\cosh(|\mathbf{k}|h)} \exp(i\mathbf{k}.\mathbf{r} \pm i\sigma t)\,dk\,dl, \tag{18}$$

the plus and minus signs referring to P_1 and P_2 respectively. Transform (18) into polar co-ordinates by setting $x = r\cos\theta$, $y = r\sin\theta$, $k = \rho\cos\eta$, $l = \rho\sin\eta$, $\chi = \rho h$, $\mu(\chi) = (\chi\tanh\chi)^{\frac{1}{2}}$ and $\mu_1(\chi) = g^{\frac{1}{2}}h^{-\frac{1}{2}}\mu(\chi)$, then

$$P_{1,2} = \frac{1}{h^2}\int_0^{\infty}\int_0^{2\pi} \frac{\hat{F}_1(\chi,\eta)}{\cosh\chi} \exp\left[i\chi\frac{r}{h}\cos(\theta-\eta) \pm it\mu_1(\chi)\right] d\eta\,d\chi, \tag{19}$$

where
$$\hat{F}_1(\chi,\eta) = F(\chi h^{-1}\cos\eta, \chi h^{-1}\sin\eta, \pm i\mu_1(\chi)).$$

Approximations to (17) are obtained by applying the method of stationary phase twice to the integrals P_1 and P_2 (van den Driessche & Braddock 1972*b*). Let

$$\phi_{\pm}(\chi,\eta) = (rh^{-1}\chi\cos(\theta-\eta))/t \pm \mu_1(\chi) \tag{20}$$

be the phase functions. The points of stationary phase are then obtained from the conditions

$$\partial\phi_{\pm}/\partial\eta = 0, \quad \partial\phi_{\pm}/\partial\chi = 0. \tag{21}$$

The first condition yields stationary points at $\eta^* = \theta$ and $\theta+\pi$ for values of θ in the range $[0,\pi]$.

Consider first the case $\eta^* = 0$; the second of conditions (21) yields

$$r/ht = \mp\,\mu_1'(\chi); \tag{22}$$

note that this is the two-dimensional equivalent of (11). Now we require $r \geqslant 0$ and $t \geqslant 0$, and ϕ_+ does not yield a stationary-phase point in χ for this particular case. Thus for $\eta^* = \theta$, a stationary point $\chi = \chi^*$ is obtained only for P_2. In a similar manner, it is found that, for $\eta^* = \theta+\pi$, there is a stationary-phase point only for P_1.

In each case, the value χ^* is obtained from the equation

$$r/t = (gh)^{\frac{1}{2}}\mu'(\chi), \tag{23}$$

and the asymptotic values

$$\left.\begin{aligned} P_1 &\approx t^{-1}\frac{2\pi\chi^*}{h^2}\frac{\hat{F}_1(\chi^*,\theta+\pi)}{\cosh(\chi^*)}|\mu_1''(\chi^*)\mu_1'(\chi^*)\chi^*|^{-\frac{1}{2}}\exp\left(-\frac{ir\chi^*}{h}+i\mu_1(\chi^*)t\right), \\ P_2 &\approx t^{-1}\frac{2\pi\chi^*}{h^2}\frac{\hat{F}_1(\chi^*,\theta)}{\cosh(\chi^*)}|\mu_1''(\chi^*)\mu_1'(\chi^*)\chi^*|^{-\frac{1}{2}}\exp\left(-\frac{ir\chi^*}{h}-i\mu_1(\chi^*)t\right) \end{aligned}\right\} \tag{24}$$

are then obtained for the integrals. Once again this has produced a sinusoidal wave of complex amplitude but the wave train decays as t^{-1}, or r^{-1}, as is easily observed from (23).

The second derivatives $\phi_{\pm\chi\chi}$ and $\phi_{\pm\eta\eta}$ are zero at $\chi^* = 0$ and the approximations (24) are not valid in this, the long-wave, limit. Further asymptotic estimates of P_1 and P_2 can be obtained by the method used by van den Driessche & Braddock (1972*b*), and give

$$P_{1,2} \approx t^{-1} 2\pi h^{-\frac{3}{2}} 3^{-\frac{1}{3}} g^{-\frac{1}{3}} \hat{F}_1(0, \eta^*). \tag{25}$$

Note that here the wave front and the wave train have the same order of magnitude, namely t^{-1}, and compare this with the one-dimensional case, where the order of magnitude of the wave front differs from the order of magnitude of the main wave system.

In a similar fashion to the one-dimensional case, the orders of the various parts of the waves can vary depending on where the function $\hat{F}_1(\chi, \eta)$ attains its zeros, if any. Suppose that the stationary point (χ^*, η^*) is at a simple zero of $\hat{F}_1(\chi, \eta)$, then

$$\begin{aligned} P_{1,2} \approx t^{-2} \frac{\pi\chi^*}{h^2} |\mu_1''(\chi^*)\mu'(\chi^*)\chi^*|^{-\frac{1}{2}} \exp\left[\mp i\left(\frac{r\chi^*}{h} - \mu_1(\chi^*)t\right)\right] \left\{\left[\frac{\partial^2}{\partial\chi^2}\left(\frac{\hat{F}_1(\chi,\eta)}{\cosh\chi}\right)\right]_{(\chi^*,\eta^*)} \right. \\ \times (-\mu_1''(\chi^*))^{-1} \exp\left(\mp\frac{i\pi}{2}\right) + \left(\frac{\partial^2}{\partial\eta^2}\hat{F}_1(\chi,\eta)\right)_{(\chi^*,\eta^*)} \\ \left. \times (\chi^*\mu_1'(\chi^*)\cosh\chi^*)^{-1} \exp\left(\pm\frac{i\pi}{2}\right)\right\}. \end{aligned} \tag{26}$$

The upper and lower signs refer to P_1 and P_2, respectively, and $\eta^* = \theta + \pi$ for P_1, and $\eta^* = \theta$ for P_2. Note that $P_{1,2}$ are both $O(t^{-2})$.

When χ^* is small, near the long-wave limit, the approximations (25) are valid only if $\hat{F}_1(0, \eta^*) \neq 0$. Further asymptotic approximations can be obtained for the case $\hat{F}_1(0, \eta^*) = 0$ by first applying the stationary-phase principle to just the η integral. This yields

$$P_{1,2} \approx \frac{1}{h^2}\left(\frac{2\pi h}{r}\right)^{\frac{1}{2}} \int_0^\infty \chi^{\frac{1}{2}} \frac{\hat{F}_1(\chi, \eta^*)}{\cosh(\chi)} \exp\left(\pm i\left(\frac{\pi}{4} - \frac{r}{h}\chi + \mu_1(\chi)t\right)\right) d\chi.$$

For $\chi^* \approx 0$, $\cosh\chi^* \approx 1$, expand $\hat{F}_1(\chi, \eta^*)$ in the form

$$\hat{F}_1(\chi, \eta^*) = \sum_{q=m}^{\infty} \frac{\chi^q}{q!} \hat{F}_1^{(q)}(0, \eta^*), \quad \text{where} \quad \hat{F}_1^{(q)} = \frac{d^q}{d\chi^q}\hat{F}_1(\chi, \eta^*).$$

The integer m gives the order of the zero of $\hat{F}_1$ at $\chi = 0$. Substituting in the above integral and applying the stationary-phase principle to the χ integral yields

$$P_{1,2} \approx \frac{(8\pi)^{\frac{1}{2}}}{h^2} \sum_{q=m}^{\infty} 6^{\frac{1}{6}(2q-3)} \hat{F}_1^{(q)} \Gamma\left(\frac{2q+3}{6}\right) (q!)^{-1} (h/g)^{\frac{1}{6}(q+3)} t^{-\frac{1}{3}q-1} \exp\left(\mp\frac{i\pi q}{6}\right). \tag{27}$$

For large values of t, only the first term of (27) is required to obtain an adequate representation of the wave. For $m = 0$, the first term of (27) reduces to (25). Also note that the order of magnitude of the wave front, i.e. $t^{-\frac{1}{3}q-1}$, steadily decreases as the degree m of the zero of $\hat{F}_1(\chi, \eta^*)$ at $\chi = 0$ increases. These last few approximations have important applications in the next section, where a more flexible model of the bottom disturbance is considered.

5. Modelling the bottom disturbance

Many attempts have been made in an effort to model the bottom disturbance causing the wave motion (see the introduction and literature cited there). These models are inadequate for two reasons: first, the actual bottom motion is not completely understood, and second, relatively simple disturbances have been used and these lack flexibility. Methods of surmounting the second difficulty will be described in detail.

Consider again the two-dimensional model which was formally solved in §2 and assume that the applied bottom velocity can be expressed in the form

$$F(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \alpha_{mnp} H_m(x)\, H_n(y)\, L_p(t) \exp\left[-\tfrac{1}{2}(x^2+y^2+t)\right]. \quad (28)$$

Here $H_m(x)$ is the Hermite polynomial of degree m, and $L_p(t)$ is the Laguerre polynomial of degree p. Note that the functions $\exp(-\frac{1}{2}x^2)\, H_m(x)$ are even or odd as m is even or odd, and form a complete set of orthogonal functions in the range $x \in (-\infty, \infty)$. For a given function $F(x,y,t)$ the constants

$$\alpha_{mnp} = (2^{m+n} m!\, n!\, \pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} F(x,y,t)\, H_m(x)\, H_n(y)\, L_p(t) \times \exp\left[-\tfrac{1}{2}(x^2+y^2+t)\right] dx\, dy\, dt. \quad (29)$$

For all realistic forms of the bottom velocity $F(x,y,t)$, the factor $2^{m+n} m!\, n!\, \pi$ will ensure that the coefficients α_{mnp} will decrease rapidly as m and n increase. Thus only a few terms of the expansions (28) are required to represent adequately the sea-floor disturbance produced by a tsunamigenic earthquake.

Many authors have previously used the assumption that the bottom velocity can be separated in the form

$$F(x,y,t) = X(x)\, Y(y)\, T(t) \quad (30)$$

(see Carrier 1971). Unfortunately most treatments of the above problems then assume explicit forms for the functions $X(x)$, $Y(y)$ and $T(t)$. Note that the form (30) is included in the expansion (28) and appears as a special class of the expansion. The expansion (28) is, then, far more general than the explicit models previously treated and, by using the result (29) to evaluate the constants α_{mnp}, can be applied to real data.

Then on taking the appropriate Fourier and Laplace transforms of (28) (Erdélyi *et al.* 1954, p. 174; Titchmarsh 1937, p. 81)

$$\hat{F}(\mathbf{k},s) = 2\pi \exp\left(-\tfrac{1}{2}\mathbf{k}.\mathbf{k}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \alpha_{mnp} (-i)^{m+n} H_m(k) H_n(l)\, (s-\tfrac{1}{2})^{p} (s+\tfrac{1}{2})^{-p-1}. \quad (31)$$

The transient motion of the (m,n,p) component of the surface disturbance corresponding to (4) arises from the pole of order $p+1$, which occurs at $s = -\frac{1}{2}$.

This transient motion is given by

$$\zeta_T(x,y,t) = \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\alpha_{mnp}\,(-i)^{m+n}(p!)^{-1}$$
$$\times\int_{-\infty}^{\infty}\Big\{\exp(-\tfrac{1}{2}\mathbf{k}.\mathbf{k}+i\mathbf{k}.\mathbf{r})H_m(k)\,H_n(l)\,[\cosh(|\mathbf{k}|h)]^{-1}$$
$$\times\Big[\lim_{s\to-\frac{1}{2}}\frac{d^p}{ds^p}\,[s(s-\tfrac{1}{2})^p\exp(st)(s^2+\sigma^2)^{-1}]\Big]\Big\}\,dk, \tag{32}$$

but this will not be explicitly evaluated.

The propagating surface disturbances are given by

$$\zeta_p(x,y,t) = \frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\alpha_{mnp}(-i)^{m+n}(P_1+P_2), \tag{33}$$

where P_1 and P_2 have been transformed into polar co-ordinates (χ,η) and are given by

$$P_{1,2} = \int_0^{\infty}\int_0^{2\pi}\chi\psi_{\pm}(\chi)\,\Theta_m^n(\chi,\eta)\exp(it\phi_{\pm})\,(\cosh\chi)^{-1}\,d\eta\,d\chi, \tag{34}$$

$\phi_{\pm}$ are given by (20) and where

$$\left.\begin{aligned}\psi_{\pm} &= \pm h^{-2}(i\mu_1(\chi)\mp\tfrac{1}{2})^p(i\mu_1(\chi)\pm\tfrac{1}{2})^{-p-1}\exp(-\tfrac{1}{2}\chi^2/h^2),\\ \Theta_m^n(\chi,\eta) &= H_m(\chi h^{-1}\cos\eta)\,H_n(\chi h^{-1}\sin\eta).\end{aligned}\right\} \tag{35}$$

The propagating surface motions have been represented as the sum of a set of normal modes of oscillation defined by the P_1 and P_2. The analysis of § 4 is applicable to the integrals (34), but the evaluation of the integrals is facilitated by first discussing the zeros of the functions $\chi\psi_{\pm}(\chi)$ and $\Theta_m^n(\chi,\eta)$. Since $\psi_{\pm}(\chi)$ does not have any real zeros, $\chi\psi_{\pm}(\chi)$ has a simple zero at $\chi = 0$. The other zeros and poles of $\psi_{\pm}(\chi)$ correspond to complex values of χ, and these do not affect the asymptotic estimation of P_1 and P_2.

The zeros $z_{n\nu}$ $(\nu = 1,2,3,\ldots,n)$ of the Hermite polynomial $H_n(z)$ are real and distinct (Sansone 1959, p. 310). The function $H_n(\chi h^{-1}\sin\eta)$, and hence $\Theta_m^n(\chi,\eta)$, is zero on the set of straight lines

$$\chi\sin\eta = hz_{n\nu}\quad(\nu = 1,2,\ldots,n), \tag{36}$$

in the χ,η plane. Similarly the function $H_m(\chi h^{-1}\cos\eta)$ is zero on the set of straight lines

$$\chi\cos\eta = hz_{m\nu}\quad(\nu = 1,2,\ldots,m). \tag{37}$$

Hence the zeros of $\Theta_m^n(\chi,\eta)$ are all simple, except where the above lines intersect. At such points there is a double zero. Now $H_n(0)$ is zero if n is odd, but is non-zero if n is even. Then $\Theta_m^n(\chi,\eta)$ may be non-zero or have a simple or a double zero at $(\chi,\eta) = (0,0)$ depending on whether m and n are even, one of m and n is odd, or both are odd.

Provided that the stationary-phase point (χ^*,η^*) is not near a zero of $\Theta_m^n(\chi,\eta)$, then P_1 and P_2 are given by (24), with

$$\hat{F}_1(\chi^*,\eta^*) = h^2\psi_{\pm}(\chi^*)\,\Theta_m^n(\chi^*,\eta^*). \tag{38}$$

	Conditions on (χ^*, η^*), m and n	Order	Equation
Main wave, χ^* not small, m, n arbitrary	(χ^*, η^*) not near a zero point	t^{-1}	(24)
	(χ^*, η^*) at a zero point	t^{-2}	(26)
Wave front, $\chi^* \approx 0$	m and n even	t^{-1}	(27)
	m odd, n even; n odd, m even	$t^{-\frac{4}{3}}$	(27)
	m odd, n odd	$t^{-\frac{5}{3}}$	(27)

TABLE 1

If (χ^*, η^*) is near a simple zero, then $P_{1,2}$, which are of order t^{-2}, are given by (26). Near the wave front at $\chi^* = 0$, $P_{1,2}$ are given by the first term in (27) and the corresponding orders of magnitude are given in table 1.

It can be seen that the nature of the wave front depends critically on the asymmetry properties of the generating region while the zeros of $\Theta_m^n(\chi, \eta)$ give rise to a complicated interference pattern within the main wave train. The space curves in polar co-ordinates (r, θ) along which the lower amplitude waves are found are obtained from (23), (36) and (37), and are given by

$$\left.\begin{aligned} r/t(gh)^{\frac{1}{2}} &= \mu'(a/\cos\eta^*), \\ \text{or} \qquad r/t(gh)^{\frac{1}{2}} &= \mu'(b/\sin\eta^*). \end{aligned}\right\} \tag{39}$$

The constants a and b are determined from the water depth h and the particular zero of $\Theta_m^n(\chi, \eta)$ which is being considered.

Figure 2 shows the space curves on which the wave amplitude is $O(t^{-2})$, for $w = a$, or b, taking the values 0·2, 0·6 and 1·0. In the figure, radial distances are in units of $r/t(gh)^{\frac{1}{2}}$, with the wave front at $r/t(gh)^{\frac{1}{2}} = 1$. The higher values of w correspond to greater values of the water depth h, and the larger zeros of the Hermite polynomials. The corresponding curves are grouped close to the origin. Lower values of w correspond to shallow water and the smaller zeros, the corresponding curves reaching out towards the wave front.

6. Conclusions

The standard techniques of integral transforms and the stationary-phase principle provide asymptotic estimates of the magnitude of a tsunami produced by sea-floor disturbances. The tsunami consists of a dispersive wave train preceded by a non-dispersive wave front travelling as a long ocean wave. The relative order of magnitude of the train and the wave front depends on the degree of symmetry or asymmetry of the bottom disturbance. In an ocean with only one horizontal co-ordinate, the wave train is $O(t^{-\frac{1}{2}})$, while the front is $O(t^{-\frac{1}{3}})$ for a general (not asymmetric) bottom disturbance. However, if the bottom disturbance is asymmetric, the front is $O(t^{-\frac{2}{3}})$, and the relative importance ot the front and the wave train is reversed.

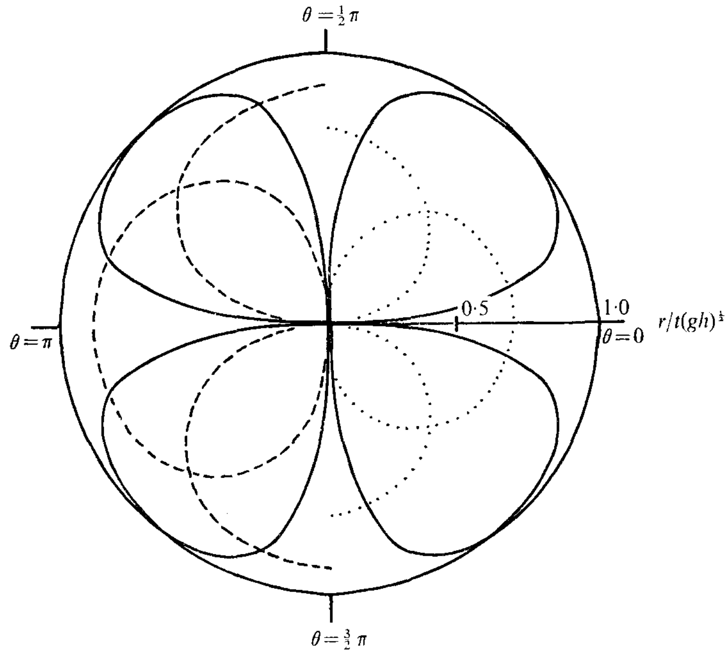

FIGURE 2. Curves in the space plane on which the wave amplitude is $O(t^{-2})$.
——, $w = 0{\cdot}2$; — — —, $w = 0{\cdot}6$; $\cdots\cdots$, $w = 1{\cdot}0$.

In an ocean with two horizontal co-ordinates, the bottom disturbance is expressed as series of orthogonal functions, and a set of normal modes of oscillation of the sea surface is produced. Again a wave train is produced and it is usually $O(t^{-1})$. However, a complicated interference pattern is apparent in each mode. The nature of the wave front again depends on the degree of asymmetry in the particular term of the series representing the bottom motion.

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